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Effective characteristics and stress concentrations in materials with self-similar microstructure

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Abstract

It is proposed to model materials with self-similar structure by a continuum sequence of continua of increasing scales each determined by its own size of the averaging volume element. The scaling is represented by power laws with the exponents determined by the microstructure, but not necessarily by the material fractal dimension. The scaling laws for tensors are shown to be always isotropic (the same exponent for all non-zero components) with the prefactors accounting for anisotropy. For materials with self-similar distributions of pores, cracks and rigid inclusions the scaling laws for elastic characteristics were determined using the differential self-consistent method. Stresses are defined in each continuum (and are measured in conventional units of stress) with the scaling law controlling the transition from one continuum to another, i.e. from one stress field to another. In the case of strong self-similarity the scaling exponent for the stress field is uniform, coincides with the one for the average (nominal) stress and is controlled by the sectional fractal dimension of the material. Within each continuum the stress concentrators—point force, dislocation, semi-infinite crack—produce conventional stress singularities. However, as the point of singularity is approached, the transition to finer continua is necessary, resulting, in some cases, in apparent non-conventional exponent of the stress increase.

Keywords: Fractal material; Scaling of elastic characteristics; Self–similar distributions; Self–similar stress concentrations; Fractal crack growth

1. Introduction

A distinctive property of many materials is the presence of internal microstructure containing sets of characteristic lengths in a range of scales (for instance, in concrete at least three markedly different scales

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are recognised, van Mier, 1992). Explicit modelling of the mechanical behaviour of such materials requires considerable computational resources. A radical simplification is achieved in the cases when the microstructure can be considered as self-similar, at least within a range of scales.

The self-similarity, i.e. the absence of a characteristic scale is typical for the microstructure at a critical stage, or at the phase transition such as the cluster structure at the percolation threshold or cellular automata structures (e.g., Bak and Tang, 1989; Chopard and Droz, 1998). Evidence of self-similar microstructures is found in metals (e.g., Mandelbrot et al., 1984), composites (e.g., Skjeltorp, 1988; Kjems and Posselt, 1988), porous rocks (e.g., Katz and Thompson, 1985), concrete and mortar (e.g., Issa and Hammad, 1993; Carpinteri, 1994; Carpinteri et al., 1995; Bazant, 1997), the loading-induced distributions of defects and dislocations (see Weiss and Marsan, 2003 and references therein). Gutenberg-Richter law is perceived as strong evidence in favour of the hypothesis of self-similarity of the Earth crust structure. Additional evidence is provided by the self-similarity in the distribution and properties of macroscopic fractures and fragments in rock masses (e.g., Hartmann, 1969; Scholz and Aviles, 1986; Gelikman and Pisarenko, 1989; Scholz, 1990; Redner, 1990; Olding, 1992; Barton and Zoback, 1992; Turcotte, 1993; Gillespie et al., 1993; Yamamoto et al., 1993; Nagahama, 1993; Dubois, 1998). Self-similar distribution of particles is also observed in the gouge of faults (see Sammis, 1997 and literature therein) and in crushed soils, Bolton and Dowell (1977). The results of the study of distribution of preferential sizes of breakups and fractures (Sadovskiy, 1983) suggest the existence of self-similarity in the distribution modes over a wide range of scales, from microns to thousands of kilometres.

Different mechanisms were suggested of either self-similarity itself or its manifestations. King and Sammis (1992) and Sammis (1997) put forward the fractal fragmentation as a mechanism of producing self-similar distributions of particles. They argue that if the particles have similar sizes, it is more likely that due to the slight size mismatch the reference particle will only be contacted and therefore loaded at two opposite points creating tensile stresses on the line connecting these points. If however the reference particle contacts particles of markedly different sizes, there will be more contacts and the load will be more distributed thus suppressing high tensile stresses. As a result, neighbouring particles of similar sizes are more prone to failure and will not survive leaving only neighbours of very different sizes. Bolton and Dowell (1977) proposed a model that explains the observed power dependence between the bulk modulus of an array of soil particles and pressure based on the Herzian-type contacts between the particles. This model is remarkable in that it predicts a power law—the conventional indicator of fractality—in a system that itself does not have to be self-similar. This is perhaps another indication that caution has to be exercised when analysing apparently fractal manifestations of natural objects. Dyskin (2000, 2001, 2002a) proposed a mechanism of developing self-similar crack distributions based on crack interaction leading to a distribution of crack sizes with the distribution function proportional to the inverse fourth power of the crack radius.

Attempts were made to analyse the deformation and fracture behaviour of fractal bodies by relating the elastic (e.g., Balankin, 1992; Cherepanov et al., 1995) and fracture (e.g., Bazant, 1993; Mosolov, 1993; Carpinteri, 1994) properties to the fractal dimension of the material i.e. to its geometric property. However, if one considers a material with ideal cracks, its fractal dimension will be exactly equal to 3, since cracks have no internal volume. Thus from the point of view of fractal approach the cracked material is indistinguishable from a solid material, while their deformation and strength properties will obviously be different (note that multiscale crack distributions of sufficiently high concentration can considerably reduce the elastic moduli, potentially to zero values). This indicates that there is no direct relationship between the mechanical scaling laws and fractal dimensions.

Balankin (1992) (see also Cherepanov et al., 1995) proposed a mechanics of fractal objects assuming: (1) external loading creates "a unique new characteristic length" and (2) in the process of deformation the material retains its self similarity. Assumption (1) apparently means that the material looses its self similarity, which contradicts assumption (2). Carpinteri (1994) determined the scaling law for nominal stress σ acting on an elementary area S of the self-similar material based on the force balance equation,

$\sigma(b)S(b) = F = \text{const}$, where b is the scale. If the area scales as $S \propto b^{D_s}$, where $D_s < 2$ is the fractal dimension of the area (sectional fractal dimension of the material), then the nominal stress scales as $\sigma \propto b^{D_s-2}$ indefinitely increasing with decreasing scale. The characteristic feature of this approach is that the self-similar nominal stress is measured in non-traditional units, $\mathrm{N\,m^{-D_s}}$. Carpenteri and Cornetti (2002) extended this approach and introduced strain having non-conventional units. Then the equations of equilibrium and the strain definition were obtained by replacing the conventional derivatives of continuum mechanics with local fractional derivatives associated with the fractal dimension of the material. In this approach the elastic modulus becomes scale independent. Another approach to modelling of overall mechanical behaviour of materials with fractal microstructure or elastic networks and lattices is to introduce scaling to their properties, for example elastic moduli (e.g., Herrmann and Roux, 1990; Zosimov and Lyamishev, 1995).

A remark should be made on the non-traditional units determined by the material fractal dimension. Firstly, if different pars of the material have different fractal dimensions the stress and strain will be expressed in non-uniform units. Secondly, the scale invariance of balance equations does not necessarily imply non-traditional units. Indeed, the force balance equation for example being rewritten as $F = \sigma(b_0)(b/b_0)^{-D_s} S(b_0)(b/b_0)^{D_s}$ allows stress $\sigma(b_0)$ and area $S(b_0)$ to have traditional units.

A popular approach to introduce fractal modelling in fracture mechanics is to consider the fracture process in the frame of the theory of percolation interpreting the macrocrack as an infinite cluster of connected defects (e.g., Sahimi and Goddard, 1986; Ostoja-Starwezki, 1989; Nishiuma et al., 1996; Chakrabati and Benguigui, 1997; Mishnaevsky, 1998) or as a result of diffusion-limited aggregation (DLA) (e.g., Herrmann, 1989; Louis and Guinea, 1989). It should however be noted that only in 2-D picture these structures actually break the plane. In real 3-D world the formation of such structures does not affect the connectedness of the body.

A number of papers considered crack propagation in fractal materials by extending the classical Fracture Mechanics concepts for straight cracks to the cracks with fractal surface. Mosolov (1991a) and Gol'dstein and Mosolov (1991) analysed the propagation of a mode I crack with fractal surface (i.e. its 2-D version, the crack with fractal profile) from the point of view of Griffith criterion. It was postulated that the crack propagates self-similarly and that the energy released during a step, l, of crack propagation should be spent on forming the fracture surface, which being a fractal of a dimension $D > 1$ has the actual length $\propto l^D$. This leads to the conclusion that the stress distribution at the crack tip has to have singularity $l^{(2-D)/2}$ Therefore if the newly developed fracture surface has the (sectional) fractal dimension $D > 1$ the crack-tip stress distribution must be weaker than the conventional square root singularity. In 3-D case, for disk-like cracks with surfaces of fractal dimension $2 < D < 3$, the singularity exponent becomes $(3-D)/2$. Gol'dstein and Mosolov (1991) also considered the case of quasi-brittle fracture assuming that the dissipative process is self-similar with its own exponent, β. In this case the stress singularity was found to be even lower with the exponent $(2-D-\beta)/2$. The arguments of this kind were used in a number of other publications (e.g., Borodich, 1992, 1994; Chakrabati and Benguigui, 1997). Further development of this approach involves the consideration of fractal microcracking (Borodich, 1997) at the crack tip self-affine fracture surface (e.g., Balankin, 1997; Weiss, 2001) and the consideration of fractal cracks in a Cosserat continuum (Yavari et al., 2002a)—a continuum whose points possess rotational degrees of freedom in addition to conventional translational ones. Xie and Sanderson (1995) and Maximov (1998) modelled dynamic propagation of fractal cracks using classical results for dynamic propagation of straight cracks.

Bazant (1997) modelled the size effect in strength by considering fracture in plates of a specified thickness, d, and of similar sizes (characteristic dimensions) D. Within a certain range of scales failure is supposed to be caused by a crack represented by a single fractal curve with dimension d_f. The apparent (smooth or projected) crack length is assumed to be proportional to the sample size, D. At larger scales, failure proceeds in a non-fractal regime as a propagation of a conventional crack with non-fractal (rectified) surface. For small cracks (lengths not exceeding the inherent defect size) and $d_f > 1$ this gives unconventional increase in strength with the sample size increase.

Mosolov and Borodich (1992) attempted to use the concept of fractal fracture surface to explain brittle fracture in compression, i.e. the experimentally observed fact that cracks can grow parallel to the direction of compression (if the crack is a cut in a homogeneous material, compression parallel to the crack does not affect it at all, so there is no mechanism of its growth). Their explanation eventually used the expression for the stress distribution at the crack tip (Eq. (5) in the cited paper) borrowed from the works which consider the crack propagation under perpendicular tensile load. Yavari et al. (2002b) extended this approach and introduced further fracture modes related to the application of normal stress in the directions parallel to the "main crack plane". [1] However, normal loading along a rough surface only creates fluctuations of the stress component normal to the "main crack plane". The mean value of this stress is zero (due to equilibrium), while the effect of the fluctuations is localised within an area of the size of asperities (in the case of fractal cracks it is the upper cut-off at most). It is not clear whether such fine scale fluctuations are able to produce the stress concentrations at the crack tip.

The formulation of fractal fracture criteria leads to fracture toughness, $K_{\mathrm{Ic}}^{\mathrm{f}}$ (Gol'dstein and Mosolov, 1991) or J-integral (e.g., Mosolov, 1991b; Rupnowski, 2001) or fracture energy (e.g., Carpinteri, 1994; Bazant, 1997) being of variable units. Their values increase with the fractal dimension, for instance $\ln K_{\mathrm{Ic}}^{\mathrm{f}}$ depends linearly on D-1. Attempts to verify this dependence experimentally (Mecholsky et al., 1989; Issa et al., 1993; Saouma and Barton, 1994) produced, according to Saouma and Barton (1994), a low goodness of fit.

It should be noted that these non-conventional stress singularities are prescribed by the energy considerations under the assumption of self-similarity of crack growth and the fractal nature of the fracture surface. The corresponding exponents have to be determined by examining the fracture surface (post-mortem examination, Saouma and Barton, 1994), no equations are formulated for calculating these exponents from the microstructural properties. This is essentially a back analysis, from the effect to the cause. A close examination of the proposed concept shows that if the loading applied to the crack is marginally smaller than the critical one (i.e., the one required for the crack propagation), the stress singularity should obviously have the same unconventional singularity whose exponent is determined by the fractal properties of the fracture surface that does not yet exist. The defenders of this approach may well argue that it is determined by the existing crack surface which, in the virtue of the postulated self-similar nature of crack growth has the same fractal characteristics as the fracture surface to be. However, the employed energy reasoning is insensitive to the existing crack surface as long as the latter does not have characteristic sizes. The same procedure could for instance be applied to the case of a conventional straight crack surface yielding the unconventional singularities, which would be in direct contradiction to the classical Fracture Mechanics. Furthermore, the assumption that the energy released during a strep of fractal crack propagation is proportional to the measure of the step was not confirmed by direct finite element simulations, Rupnowski (2001). Interestingly, the correspondence between the computed values of J integral and the measure is the closer the smoother the crack trajectory. Apparently, they will coincide for a straight crack, as implied by the classical Fracture Mechanics.

The above approaches are concerned with scaling of stress, strain or elastic moduli. The use of such quantities however presumes the existence of a continuum, which contradicts the notion of fractal materials as being inherently discontinuous. Dyskin et al. (1992) proposed a way to overcome this difficulty by introducing several equivalent continua each responsible for its own scale. Dyskin (1999a,b) extended this approach to self-similar structures and then modelled the crack-induced stress concentrations (Dyskin, 2002b) and the mechanical behaviour of the Earth's crust (Dyskin, 2004). The present paper develops these concepts further by considering scaling laws for materials with different types of inhomogeneities and stress concentrators.

[1] We introduced this term to describe the general orientation of the fractal crack sketched by Yavari et al. (2002b) in diagrams.

2. Multiscale continuum mechanics

The use of conventional quantities of mechanics-stress, strain, moduli etc. require the introduction of a continuous medium. Since there are no continuous materials in nature, the continuum approximation is based on the introduction of the volume element (elementary volume or representative volume element, e.g., Nemat-Nasser and Hori, 1993) of a certain size, H, satisfying the following double inequality (e.g., Scipio, 1967; Batchelor, 1974; Hunter, 1976; Dyskin et al., 1992; Krajcinovic, 1996; Jeronimidis, 2000):

$$l \ll H \ll L \tag{1}$$

where l is the characteristic size of the material microstructure (e.g., the grain or defect size, the distance between the microstructural elements, etc. If a number of sizes are involved, then l is assigned to the largest one), L is the characteristic length of the variations of the external fields (e.g., the characteristic size of the area under consideration, the wave length, etc.). The continuum mechanics quantities (macroscopic in the spirit of the inequality (1)) essentially represent the microscopic quantities averaged over the volume elements of size H, V_H (e.g., Batchelor, 1974). Thus, if σ_{ij}^l and ε_{ij}^l are stress and strain fields (not necessarily continuous) at scale l, the stress and strain fields at scale H are given by

$$\sigma_{ij}^H(\mathbf{x}) = \frac{1}{V_H}\int_{V_H} \sigma_{ij}^l(\mathbf{x}+\mathbf{t})\mathrm{d}V_t, \quad \varepsilon_{ij}^H(\mathbf{x}) = \frac{1}{V_H}\int_{V_H} \varepsilon_{ij}^l(\mathbf{x}+\mathbf{t})\mathrm{d}V_t \tag{2}$$

As prescribed by the first part of inequality (1), the volume elements V_H are supposed to be much larger than the microstructural elements such that they are representative of the microstructure (the structure at scale l). On the other hand, the second part of the inequality (1) makes the volume element size asymptotically infinitesimal as compared to the characteristic dimensions of the region under study such that the variations of the studied quantities over the volume element size can be neglected (e.g., Scipio, 1967; Hunter, 1976). Thus, in the macroscopic description associated with the size L the material is replaced by a continuum, while the volume element size H plays the role of the resolution of the continuum model (e.g., Krajcinovic, 1996) when the results are interpreted in terms of the original material with the microstructure. Consequently the material is replaced with a continuum whose behaviour models the behaviour of the original material at the scale H (Fig. 1). Correspondingly, the constitutive properties of the continuum will specify the relationship between the averages over the volume elements the parameters of this relationship being the effective characteristics.

Usually in classical continua, the size H does not explicitly appear in the equations, however it is to be reckoned with when the results of the modelling are interpreted in terms of the behaviour of the original material. It might also enter through, for example, fracture toughness (in fracture mechanics considerations). It is present in Cosserat, high gradient or non-local continua. We will call the corresponding equivalent continuum, the H-continuum.

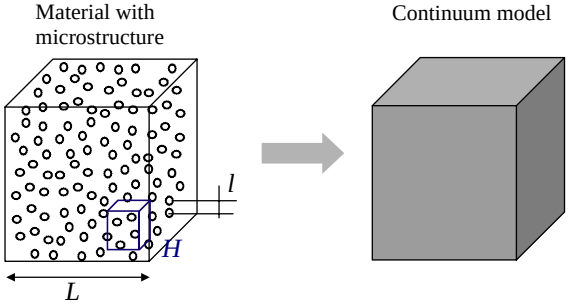

Fig. 1. Continuum modelling of materials with microstructure by the introduction of an averaging volume element.

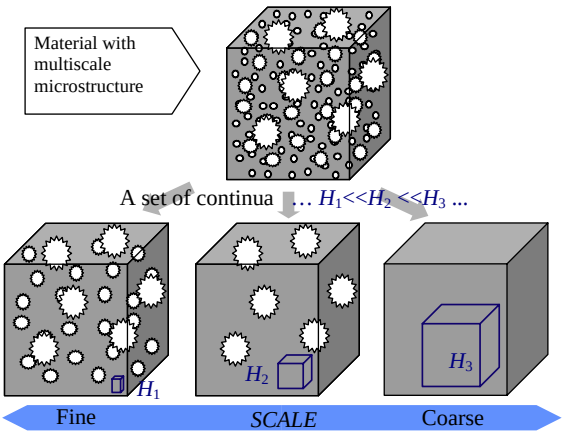

Fig. 2. Multiscale modelling: material with multiscale structure is modelled by a sequence of continua characterised by different averaging size.

There are a number of cases when materials contain a multiscale hierarchical structure ranging up to the scale of consideration, L such that one cannot choose a volume element size H satisfying (1). The situations of this kind arise when the scale L is relatively small (e.g., in microelectronics). At larger scales these structures include biomaterials (Jeronimidis, 2000) and the Earth's crust (e.g., Jaeger, 1979; Cuisiat and Haimson, 1992). Such materials can be modelled by a set of continuous media each characterised by its own scale. Formally it can be done by introducing a set of volume elements of different sizes (e.g., Dyskin et al., 1992; Jeronimidis, 2000). Volume elements of each size, H, define a continuous medium of scale H, the state of the medium being determined by stresses and strains averaged over the H-volume elements (Fig. 2). *The H-continuum replaces the material with a material possessing modified microstructure in which only those microstructural elements present that have characteristics sizes greater than H.*

In general, the relationship between the continua is complex, especially in the case when there is not enough clearance between the scales (condition $l \ll L$ is not satisfied) such that each continuum cannot be obtained by averaging the state variables of smaller scales as prescribed by Eq. (2). There is however a case when this relationship can easily be obtained. This is the case of materials with self-similar microstructure, i.e. the macrostructure that does not posses a characteristic length. Such a microstructure is indeed an abstract concept which can only be used to model some real materials at most within a certain range of scales between the lower and upper cut-offs (see Dyskin, 2004 for other restrictions). In the limit $H \to 0$ one could obtain a description of a fractal material of the kind proposed by Panagouli (1997) and Panagiotopoulos and Panagouli (1997). However, in many cases, especially in view of the presence of the lower cut-off, a whole range of continua is needed as described in the following section.

3. Continuum mechanics of materials with self-similar microstructure

Let the material's microstructure be self-similar such that there is no characteristic size in the microstructure. This means that there is no preferential choice in values of H for the modelling continua. Therefore, such a material should be modelled by a continuous set of continua with the volume element sizes, H assuming all values, Fig. 3. In this case, all continuum quantities should also be functions of scale, [2] H.

[2] Rodionov et al. (1989) suggested to associate the scale with the fifth dimension added to the conventional three spatial and a temporal ones. It should be noted that in the present analysis the fifth co-ordinate is so-far an index pointing to the modelling continua.

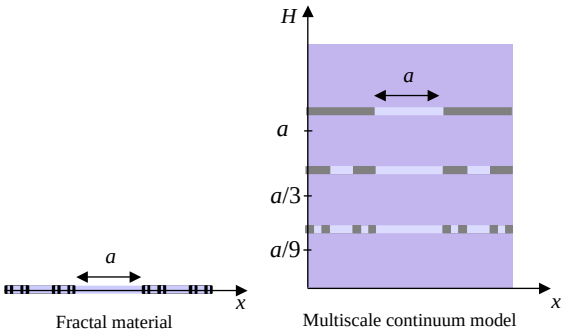

Fig. 3. The concept of multiscale continuum modelling. The fractal material is illustrated by a Cantor set. It is modelled by a continuum of continua such that on top of the special coordinates (the x-coordinate in this illustration) a new coordinate H (scale) is introduced. At a certain scale only the objects with the sizes greater than that scale can be recognised. Three such scales, a, $a/3$ and $a/9$ are shown.

Among all quantities the simplest are the uniform ones, i.e. the quantities that are independent of spatial coordinates. As an example, consider an H-continuum and remove all the features of larger scales. If this continuum is then homogeneous (i.e. the distributions of the smaller scale features are statistically homogeneous), the characteristics of the H-continuum will only be functions of H. The quantities that are positive functions of scale (or, generally, functions of H of constant sign) only are represented by power functions, $f(H) \propto H^{\alpha}$, according to the general theorem (e.g., Barenblatt and Botvina, 1980; Gelikman and Pisarenko, 1989; Zosimov and Lyamishev, 1995). [3] This implies that in the case of materials with self-similar microstructure the transition from one continuum to another is determined by power laws.

In particular, if the modelling continua are linearly elastic, the case considered hereafter, the 6×6 symmetric matrix of general anisotropic moduli or compliances should have scaled as $C_{ij}(H) \propto H^{\alpha}_{ij}$ where $i, j = 1,\ldots,6$ and α_{ij} could be thought of as 21 generally independent constants. [4] However, the elastic moduli form a tensor and it is this tensorial property that is so restrictive that reduces the number of exponents to one, as explained in the following section.

4. Scaling laws for tensorial characteristics

The fact that the tensor components have to change with the coordinate rotation in accordance with a certain rule imposes further restrictions on the scaling laws. Consider, for the sake of simplicity a Cartesian coordinate set (x_1, x_2, x_3). Suppose a tensor $T_{ijk\ldots}(H) \propto H^{\alpha_{ijk\ldots}}$ where $i, j, k_{\ldots} = 1, \ldots, 3$. When the coordinate set is rotated with a matrix r_{ij}, the new components of the tensor will be. [5]

$$T'_{ijk\ldots} = r_{il} r_{jm} r_{kn} \ldots T_{lmn\ldots}, \quad T'_{ijk\ldots} = t'_{ijk\ldots} H^{a_{ijk\ldots}}, \quad T_{lmn\ldots} = t_{ijk\ldots} H^{\alpha_{lmn\ldots}} \tag{3}$$

[3] Usually a condition of continuous differentiability is imposed on $f(H)$. Gelikman and Pisarenko (1989) relaxed this condition by demonstrating that any function of H only (not depending on a parameter of length) bounded on an interval $[H_1, H_2]$ is a power function. Consequently, it is infinitely differentiable for all $H > 0$.

[4] The matrix of elastic moduli is symmetric and positive definite which already poses some restrictions on the exponents. Indeed, the entries of such a matrix can be represented as $C_{ij}(H) = \mu_{ij}(C_i C_j)^{1/2}$, $i, j = 1, \ldots, 6$, Tshernykh (1988), where the generalised Poisson's ratios, μ_{ij}, are bounded. Therefore, μ_{ij}, are trivially scaled the corresponding exponents being equal to zero; only the moduli, C_i, are scaled non-trivially. This implies that out of 21 elastic constants of general anisotropic tensor of elastic moduli or compliances not more than 6 can scale independently. Similar conclusion can be made about the compliancy matrix.

[5] Hereafter the prefactors will be denoted by low case letters.

Here summation over repeated indices is presumed.

The tensorial equation (3) establishes a linear relationship between power functions. However, power functions with different exponents are linearly independent (Appendix A). Consequently, the following proposition holds.

Proposition. *For a tensor that depends on scale, H, as the only variable of units of length, all non-zero components must scale with the same exponent, i.e.*

$$\text{Either } T_{ijk\ldots} = 0 \quad \text{or } T_{ijk\ldots} \propto H^{\alpha} \tag{4}$$

where α is a constant common for all $i,j,k,\ldots$

Proof. Suppose the exponents in (4) assume p different values, $\alpha_1, \ldots, \alpha_p$. Then by grouping the terms with the same powers one gets

$$\sum_{q=1}^{p} C_q H^{\alpha_q} = 0 \tag{5}$$

where C_q are the sums of the corresponding terms $r_{il}r_{jm}r_{kn}\ldots t_{lmn\ldots}$; one of these sums contains (or just consists of) the component $-t'_{ijk\ldots}$. Because the power functions with different exponents are linearly independent (Appendix A),

$$C_q = 0, \ q = 1, \ldots, p \tag{6}$$

When another coordinate rotation, r, is chosen, only terms $-t'_{ijk\ldots}$ could, in principle, change the power and either migrate to another group or form a new one. Since the number of groups, p, is finite, while the number of possible different rotations r is infinite, one can always find a group that does not contain terms $-t'_{ijk\ldots}$ for sufficiently large number of different rotations. Thus, one obtains a homogeneous system of equations

$$r_{il}r_{im}r_{kn}\ldots t_{lmn\ldots} = 0 \tag{7}$$

where the summation is presumed over the values of indices l, m, n, ... that belong to the group in question. Since the components of the rotation matrix are linearly independent functions (they are combinations of sines and cosines), $t_{lmn\ldots} = 0$.

Continuing in this way, one can eliminate all groups except the one containing $-t'_{ijk\ldots}$. Therefore, all non-zero components of the tensor will have the same power, the one that corresponds to this last group. This finalises the proof. □

The proposition implies that all components of a tensor must have the same scaling law, which greatly reduces the possible set of scaling laws that could be associated with a given self-similar structure. This is a general statement applicable to all tensorial properties and quantities independent of the type of continuum or its rheology. The following section explores this property in the application to the scaling of elastic characteristics.

5. Scaling of elastic moduli and compliances

The restrictions imposed by the tensorial property (Proposition) imply that the tensors of elastic moduli and compliances can be written, in both matrix and tensorial forms, as follows

$$C_{ij}(H) = c_{ij}H^{\alpha}, \quad A_{ij}(H) = a_{ij}H^{\beta}, \quad i,j = 1,\ldots,6 \tag{8}$$

$$C_{ijkl}(H) = c_{ijkl}H^{\alpha}, \quad A_{ijkl}(H) = a_{ijkl}H^{\beta}, \quad i,j,k,l = 1,2,3 \tag{9}$$

where C_{ijkl} is the tensor of elastic moduli, A_{ijkl} is the tensor of compliances and, obviously, $\alpha = -\beta$.

Therefore, the tensors of elastic moduli and compliances must scale isotropically. This property is independent of the microstructure meaning that no matter what the anisotropy of the material is the scaling must be isotropic. The anisotropy is accounted for by the prefactors, c_{ijkl} and a_{ijkl} of the power functions (9).

The particular values of α, β, c_{ijkl} and a_{ijkl} depend on the material microstructure. In order to obtain a system of equations for the determination of these parameters, consider a transition from a scale H to the scale $H + \mathrm{d}H$. Since this transition involves the inclusion of new structural elements in the definition of the equivalent continuum, its compliances and moduli will be incremented.

$$A_{ijkl}(H + \mathrm{d}H) = A_{ijkl}(H) + \Delta A_{ijkl}(H)\,\mathrm{d}H \tag{10}$$

The tensor increment, $\Delta A_{ijkl}(H)$, is determined by both the new structural elements and their interaction with the existing elements, however its dependence of the scale should again be expressed by a power law:

$$\Delta A_{ijkl}(H) = \Delta a_{ijkl}H^{\gamma} \tag{11}$$

Taking into account that $A_{ijkl}(H + \mathrm{d}H)$- $A_{ijkl}(H) = \beta a_{ijkl}\, H^{\beta-1}\mathrm{d}H$ one can write $\beta a_{ijkl}H^{\beta-1} = \Delta a_{ijkl}H^{\gamma}$. From here, $\gamma = \beta - 1$ and

$$\beta a_{ijkl} = \Delta a_{ijkl} \tag{12}$$

or in matrix form

$$\beta a_{ij} = \Delta a_{ij}, \quad i,j = 1,\ldots,6 \tag{13}$$

This is generally a system of 21 equations for 22 unknowns, a_{11} , …, a_{66} and β (here the symmetry of matrix a_{ij} is taken into account). Since the prefactors for both compliances and the increments have the same units, one of the compliance prefactors can be chosen arbitrarily, while the others and the exponent can be found from (13). In the following sections this system will be further simplified for a special case of materials with self-similar distributions of isolated inhomogeneities (voids, cracks, defects, inclusions, etc.) using the differential self-similar scheme and solved for some special cases.

A similar system can obviously be written in terms of elastic moduli

$$\beta c_{ij} = \Delta c_{ij}, \quad i,j = 1,\ldots,6 \tag{14}$$

6. Self-similar distributions of inhomogeneities

6.1. General form

Consider a material containing self-similar distribution of inhomogeneities (voids, cracks, defects and inclusions). Generally, the self-similarity means that the distribution law is expressed by a power function. For instance the probability density function (“distribution function” in the statistical physics’ terminology) has the form

$$f(R) = \frac{w}{R^m} \tag{15}$$

where R is the characteristic size of the heterogeneity, w is a normalising constant.

The usual normalisation

$$\int_0^{\infty} f(R)\mathrm{d}R = 1$$

leads to divergent integrals for any m. Therefore a self-similar distribution can only be viewed as an approximation of the real one that ranges between lower and upper cut-offs, $R_{\min}$ and $R_{\max}$. Consequently, the normalisation factor should depend on the cut-offs. We will determine it by assuming that the total concentration of inhomogeneities, v_t is specified. Then

$$\int_{R_{\min}}^{R_{\max}} R^3 f(R)\mathrm{d}R = v_t \tag{16}$$

With this normalisation, $f(R)\mathrm{d}R$ is the number of inhomogeneities of sizes from R to $\mathrm{d}R$ per unit volume.

Obviously, $w\to 0$ as $R_{\max}/R_{\min}\to\infty$. Thus the self-similarity can be regarded as an asymptotic property in the limit $R_{\max}/R_{\min}\to\infty$ with w being asymptotically very small. It will also be assumed that the total concentration of inhomogeneities, v_t remains constant in this limit transition.

6.2. Self-similarity in narrow sense. Wide distribution of sizes

Eq. (15) gives the general form of the self-similar distribution, however, the effective characteristics are controlled by the dimensionless concentration of inhomogeneities, $N\langle R^3\rangle$ (e.g., Salganik, 1973), where N is the total number of inhomogeneities per unit volume and $\langle\cdot\rangle$ stands for averaging. One can suggest the definition of *self-similarity in narrow sense* as a distribution in which the concentration of inhomogeneities is the same at every scale. Consider a range of scales between R and nR, where $n>1$ is a constant. The concentration of inhomogeneities in this range is

$$v(R,nR) = N\int_R^{nR} R^3 f(R)\mathrm{d}R = \begin{cases} \dfrac{Nw}{4-m}R^{4-m}(n^{4-m}-1) & \text{when} \quad m\neq 4 \\ Nw\ln n & \text{when} \quad m=4 \end{cases} \tag{17}$$

Obviously, the self-similarity in narrow sense is satisfied only for $m=4$. Consequently, only the distribution

$$f(R) = \frac{w}{R^4}, \quad w = \frac{v_t}{\ln R_{\max}/R_{\min}} \tag{18}$$

is considered hereafter.

Distribution (18) has an important property which underpins the suggested asymptotic procedure for determining the scaling lays for the effective moduli. Consider the probability, $P(n)$, that in a vicinity of an inhomogeneity of size R, i.e. a region of size proportional to R, there are inhomogeneities of smaller sizes, say from R/n to R, where $n>1$:

$$P(n)\sim R^3\int_{R/n}^{R} f(R)\mathrm{d}R = \begin{cases} \dfrac{w}{m-1}R^{4-m}(n^{m-1}-1) & \text{when} \quad m\neq 1 \\ wR^3\ln n & \text{when} \quad m=1 \end{cases} \tag{19}$$

When $m=4$, which corresponds to distribution (18), $P(n)\sim w(n^3-1)$, i.e. it does not depend on the inhomogeneity size, R. Since (18) represents real distributions only asymptotically as $R_{\max}/R_{\min}\to\infty$ i.e. as $w\to 0$ ($v_t=\text{const}$), for any n the value of w can be chosen small enough to make the probability negligible for any inhomogeneity size. On the other hand for $m\neq 4$ the probability depends on R and therefore cannot be made negligible for either large ($m<4$) or small ($m>4$) values of R.

This property suggests that for any inhomogeneity the probability to find nearby an inhomogeneity of a similar size is asymptotically negligible; as $w\to 0$ ($v_t=\text{const}$) only inhomogeneities of greatly different sizes can be found there. Therefore, for any H-continuum the concentration of inhomogeneities for which the inequality (1) is violated is infinitesimal and hence can be neglected. Correspondingly, averaging (2) can be used to define the continua and with it the conventional theory of effective characteristics can be employed to derive the scaling laws.

Furthermore, the interaction between the inhomogeneities of similar sizes can asymptotically be neglected; only interaction between inhomogeneities of very different sizes should be taken into account. In

essence, the limiting transition $w \to 0$ ensures infinitesimal concentration of inhomogeneities at each scale implying that their interaction can be neglected, with finite total concentration such that the interaction of inhomogeneities is reduced to the inter-scale interaction. This property of the distribution is directly equivalent to the assumption of *wide distribution of sizes* made by Salganik (1973), which makes it possible to use the differential self-consistent method for calculating the effective characteristics.

7. Effective characteristics of materials with self-similar (in the narrow sense) distributions of inhomogeneities

7.1. General equations

The differential self-consistent method (Salganik, 1973) is based on the assumption that inhomogeneities of equal size do not interact directly, while the interacting ones are very different in size. The interaction is taken into account by considering each inhomogeneity alone in an equivalent continuum with the effective characteristics determined by all inhomogeneities of smaller sizes. Thus the total values of effective characteristics are obtained incrementally by adding at each step the contribution of non-interacting inhomogeneities considered in an equivalent continuum determined by the inhomogeneities already considered at previous steps. Dyskin (2002a) showed that in the asymptotics of *wide distribution of sizes* of inhomogeneities the differential self-consistent method gives accurate values of the effective characteristics.

Consider Eq. (13). According to the differential self-consistent method, the compliancy increment ΔA_{ij}, $i,j = 1, \ldots, 6$, at each scale is determined by the contribution of non-interacting inhomogeneities considered in an effective continuum. This contribution is proportional to the concentration of the group of inhomogeneities at hand, $w\mathrm{d}H/H$ since the inhomogeneities of the same scale do not interact due to their low concentration. System (13) can then be rewritten in the form

$$A_{ij}(H + \mathrm{d}H) = A_{ij}(H) + wS_{ij}(A_{11}, \ldots, A_{66}, A^1_{11}, \ldots, A^1_{66}, \ldots, A^k_{11}, \ldots, A^k_{66})\frac{\mathrm{d}H}{H} \tag{20}$$

or, using (9),

$$\beta a_{ij} H^\beta = wS_{ij}(A_{11}, \ldots, A_{66}, A^1_{11}, \ldots, A^1_{66}, \ldots, A^k_{11}, \ldots, A^k_{66}) \tag{21}$$

Here $A^1_{ij}, \ldots, A^k_{ij}$ are the compliances associated with the inhomogeneities, function S_{ij} is specific for the given type of inhomogeneities and given distribution of their parameters. From the dimension considerations it is clear that S_{ij} is a homogeneous function of the first degree with respect to the arguments $A_{ij}, A^1_{ij}, \ldots, A^k_{ij}$.

There are two special cases for which Eq. (21) can be simplified. The first case is the case of homogeneous inclusions made of one type of material, such that $k = 1$. The second case is when the parameters $A^1_{ij}, \ldots, A^k_{ij}$ are absent in (21). This is the case of voids (pores or cracks) and rigid inclusions.

7.1.1. Elastic inclusions made of the same material

In this case Eq. (21) reads

$$\beta a_{ij} H^\beta = wS_{ij}(A_{11}, \ldots, A_{66}, A^1_{11}, \ldots, A^1_{66}) \tag{22}$$

Using (8) and the fact that S_{ij} is a homogeneous function of the first degree, Eq. (22) can be rewritten as follows

$$\beta a_{ij} = wS_{ij}(a_{11}, \ldots, a_{66}, A^1_{11}H^{-\beta}, \ldots, A^1_{66}H^{-\beta}) \tag{23}$$

Since the left part of (23) is independent of H and not all components of A^1_{ij} are equal to zero, the exponent $\beta = 0$. This results in the following equation

$$0 = S_{ij}(a_{11}, \ldots, a_{66}, A^1_{11}, \ldots, A^1_{66}) \tag{24}$$

Furthermore, because the inclusions placed in the continuum with the same characteristics as the material of inclusions do not produce any contribution to the effective characteristics the obvious solution of (24) is

$$A_{ij} = a_{ij} = A^1_{ij} \tag{25}$$

The obtained result is transparent: the self-similar distribution of inclusions does not leave place for any considerable amount of the original material such that the effective characteristics coincide with the characteristics of the material of inclusions.

7.1.2. Voids and rigid inclusions

Taking into account the power dependence of A_{ij} on H and the fact that according to the dimension analysis S_{ij} is a homogeneous function of the first degree, system (13) can be rewritten in the following form

$$\beta a_{ij} = wS_{ij}(a_{11}, a_{12}, \ldots, a_{66}), \quad i,j = 1,\ldots,6 \tag{26}$$

The corresponding system for elastic moduli reads

$$\alpha c_{ij} = w\Lambda_{ij}(c_{11}, c_{12}, \ldots, c_{66}), \quad i,j = 1,\ldots,6 \tag{27}$$

where the function Λ_{ij} represents the contribution of inhomogeneities to the elastic moduli at each step of the self-consistent method. It is also a homogeneous function of the first degree. Obviously, these systems can also be expressed in terms of 4th rank tensors of compliances or elastic moduli.

Remarks

1. Functions wS_{ij} and $w\Lambda_{ij}$ are essentially the traditional solutions for the effective characteristics in the case of non-interacting inhomogeneities considered in a continuum of a corresponding type of anisotropy the effective characteristics being replaced by prefactors and the concentration being replaced by w.
2. In isotropic case and some types of anisotropy, engineering constants are introduced (e.g., Lekhnitskii, 1968) that correspond to Young's moduli and Poisson's ratios associated with loading in different directions. Since there is a one-to-one correspondence between the components of tensors of effective characteristics and the engineering constants, systems (26) or (27) can be rewritten in terms of engineering constants. (It should be noted that because Poisson's ratios, ν_{ij}, are bounded, their scaling exponents vanish.) In some cases this representation is convenient and will be used below.
3. Systems (26) and (27) resemble the systems that would be obtained if the symmetric self-consistent method (e.g., Budiansky, 1965, 1976) is formally used for a material with zero compliances of moduli and then the effective characteristics are replaced with the prefactors and concentrations are replaced with w/β or w/α.

In the following subsections few examples of solution of system (26) or (27) will be presented.

7.2. Material with spherical pores

Consider a material with self-similar distribution of spherical pores (distribution (18)), where R is the pore radius (larger pores are superimposed on the smaller ones). For an isotropic material with non-interacting pores the effective moduli E, ν, can be found using the formula (e.g., Mackenzie, 1950; Vavakin and Salganik, 1978)

$$\begin{cases} E = E_{\mathrm{m}}\left[1 - 2\pi\dfrac{(1-\nu_{\mathrm{m}})(9+5\nu_{\mathrm{m}})}{7-5\nu_{\mathrm{m}}}v\right] \\ \nu = \nu_{\mathrm{m}} + 2\pi\dfrac{(1-\nu_{\mathrm{m}}^2)(1-5\nu_{\mathrm{m}})}{7-5\nu_{\mathrm{m}}}v \end{cases} \tag{28}$$

where E_m, ν_m, are the Young's modulus and Poisson's ratio of the original material, $v = N\langle R^3\rangle$ is the dimensionless concentration of pores (the porosity $p = 4\pi v/3$). Extracting the factors at the concentration, replacing the latter with w and the Young's modulus with prefactor, e, and taking into account that exponent for the Poisson's ratio vanishes one obtains the following system of scaling equations

$$\begin{cases} \alpha e = -2\pi e \dfrac{(1-\nu)(9+5\nu)}{7-5\nu} w \\ 0 = 2\pi \dfrac{(1-\nu_m^2)(1-5\nu)}{7-5\nu} w \end{cases} \tag{29}$$

Since $w \neq 0$ the only solution of (29) is

$$\nu = 0.2, \quad E = eH^{\alpha}, \quad \alpha = -\frac{8}{3}\pi w \tag{30}$$

where e is an arbitrary normalising factor.

7.3. Material with rigid spherical inclusions

Consider a material with self-similar distribution of spherical inclusions made of an infinitely stiff material. For an isotropic material with non-interacting rigid inclusions the effective moduli E, ν, can be found using the formula (e.g., Vavakin and Salganik, 1978)

$$\begin{cases} E = E_m\left[1 + 4\pi \dfrac{(1-\nu_m)(3-\nu_m+5\nu_m^2)}{(1+\nu_m)(4-5\nu_m)} v\right] \\ \nu = \nu_m + 2\pi \dfrac{(1-\nu_m)(1-2\nu_m)(1-5\nu_m)}{(4-5\nu_m)} v \end{cases} \tag{31}$$

where E_m, ν_m, are the Young's modulus and Poisson's ratio of the original material, $v = N\langle R^3\rangle$ is the dimensionless concentration of inclusions. Using the same procedure as for the case of pores one obtains the following system of equations

$$\begin{cases} \alpha e = 4\pi e w \dfrac{(1-\nu)(3-\nu+5\nu^2)}{(1+\nu)(4-5\nu)} \\ 0 = 2\pi w \dfrac{(1-\nu)(1-2\nu)(1-5\nu)}{4-5\nu} \end{cases} \tag{32}$$

Similarly, the only solution of (32) is

$$\nu = 0.2, \quad E = eH^{\alpha}, \quad \alpha = (4/3)\pi w \tag{33}$$

where e is an arbitrary normalising factor.

7.4. Material with randomly (isotropically) oriented elliptical cracks

This is the case of a material containing self-similar distribution of cracks that are infinitesimal cats of elliptical shape. The cracks are assumed to have the same aspect ratio, k_a, and to be randomly (isotropically) oriented. In the case of isotropic material with non-interacting cracks the effective moduli E, ν, can be found as follows (e.g., Salganik, 1973)

$$\begin{cases} E = E_m\left\{1 - \dfrac{8}{45}\pi(1-\nu_m^2)[C_1(\nu_m) + C_2(\nu_m) + 3C_3(\nu_m)]v\right\} \\ \nu = \nu_m - \dfrac{4}{45}\pi(1-\nu_m^2)\{2(1+3\nu_m)C_3(\nu_m) - [C_1(\nu_m) + C_2(\nu_m)](1-2\nu_m)\}v \end{cases} \tag{34}$$

$$C_1(v) = \frac{k^2 k_a^2}{(k^2 - v)E(k) + v k_a^2 K(k)}, \quad C_2(v) = \frac{k^2 k_a^2}{(k^2 + v k_a^2)E(k) - v k_a^2 K(k)}$$

$$C_3(v) = \frac{k_a^2}{E(k)}, \quad k^2 = 1 - k_a^2$$

Here E_m, v_m, are the Young's modulus and Poisson's ratio of the original material, $v = N\langle R^3\rangle$ is the dimensionless concentration of cracks, R is the major semi-axis of the ellipses, $K(k)$ and $E(k)$ are the elliptical integrals of the first and second kind respectively. The scaling equations then assume the form

$$\begin{cases} \alpha = -\dfrac{8\pi}{45} w(1 - v^2)[C_1(v) + C_2(v) + 3C_3(v)] \\ 0 = 2(1 + 3v)C_3(v) - [C_1(v) + C_2(v)](1 - 2v) \end{cases} \tag{35}$$

By direct substitution one can see that one of the solutions of (35) is

$$v = 0, \quad \alpha = \frac{8\pi}{9} w C_3\ (v) \tag{36}$$

It is shown in Appendix B that Eq. (35) has no other solutions. Finally, the scaling law can be obtained in the following form

$$v = 0, \quad E = eH^{\alpha}, \quad \alpha = -\frac{8\pi}{9}\frac{k_a}{E(k)} w \tag{37}$$

For a special case of disk-like cracks, $k_a = 1$, $k = 0$, $E(k) = \pi/2$. Therefore

$$\alpha_{disk} = -\frac{16}{9} w \tag{38}$$

7.5. Self-similar distributions in two dimensional case

In the 2-D case the self-similar distribution in narrow sense will have the form

$$f(l) = \frac{\omega}{l^3} \tag{39}$$

where l is the characteristic size of the inhomogeneities, ω is a normalising constant. It is assumed that in real materials this size ranges between l_{min} and l_{max} and the normalisation based on a given total 2-D concentration Ω_t is used:

$$\Omega_t = \int_{l_{min}}^{l_{max}} l^2 f(l) \mathrm{d}l, \quad \omega = \frac{\Omega_t}{\ln l_{max}/l_{min}} \tag{40}$$

It should be kept in mind that in this analysis the plane strain approximation corresponds to a body with the length in the third dimension considerably exceeding l_{max}, while the plane stress approximation corresponds to a plate with the thickness considerably smaller than l_{min}.

Similarly to the three-dimensional case we will consider the asymptotics $\omega \to 0$ ($\Omega_t = \text{const}$). This will ensure that the inhomogeneities of the same scale and in infinitesimal concentration and hence do not interact directly.

7.6. Plane with circular holes

For a plane with non-interacting pores occupying a fraction $p \ll 1$ of the total area the effective moduli in the approximation linear with respect to p read (e.g., Kachanov, 1993)

$$E = E_{\mathrm{m}}[1-3p], \quad v = \frac{1}{3} + \left(v_{\mathrm{m}} - \frac{1}{3}\right)[1-3p] \tag{41}$$

Representing the porosity via concentration $\Omega = N\langle l^2\rangle$, where N is the number of pores per unit area, $\langle l^2\rangle$ is the average square of the pore diameter, one has for the pore-determined increments of the moduli, ΔE and Δv:

$$\Delta E = -3\pi\Omega E_{\mathrm{m}}, \quad \Delta v = -3\pi\Omega\left(v_{\mathrm{m}} - \frac{1}{3}\right) \tag{42}$$

Taking into account that the scaling exponent for the Poisson's ratio is zero (since the Poisson's ratio is bounded) Eq. (27) gives $\alpha e = -3\pi\omega e, 0 = -3\pi\omega(v - \frac{1}{3})$ with the solution

$$v = \frac{1}{3}, \quad E = eH^{\alpha}, \quad \alpha = 3\pi\omega \tag{43}$$

7.7. Plane with randomly oriented cracks

Plane with randomly oriented cracks (straight cuts) is isotropic. The expressions for effective Young's modulus, E, and Poisson's ratio, v, in the case of non-interacting cracks are (e.g., Bristow, 1960; Salganik, 1973):

$$E = E_{\mathrm{m}}\left[1 - \frac{\pi}{4}\Omega\right], \quad v = v_{\mathrm{m}}\left[1 - \frac{\pi}{4}\Omega\right] \tag{44}$$

where Ω is the crack concentration, E_{m} and v_{m} are the Young's modulus and Poisson's ratio of the material. From here the corresponding components of tensor Λ_{ij} are $\Lambda_E = -E\pi/4$, $\Lambda_v = -v\pi/4$. Then, the scaling equations are: $\alpha e = -\frac{\pi}{4}\omega e$, $\alpha v = -\frac{\pi}{4}\omega v$. Since according to the first of these equations $\alpha \neq 0$, the only solution of the second equation reads $v = 0$. This yields

$$v = 0, \quad E = eH^{\alpha}, \quad \alpha = -\pi\omega/4 \tag{45}$$

7.8. Plane with two mutually orthogonal sets of cracks

Consider now a 2-D problem for a plane with two mutually orthogonal sets of cracks, Fig. 4. It will be assumed that the set of cracks perpendicular to the x_i axis is characterised by the distribution ω_i/l^3 such that the total distribution is ω/l^3 with the concentration factor $\omega = \omega_1 + \omega_2$.

In the asymptotics $\omega \to 0$ ($\Omega_t = \mathrm{const}$) the cracks of both systems of the same scale are well apart from each other such that in a vicinity of a crack only very small cracks can be found. Therefore, the instances when the cracks from these mutually orthogonal sets of cracks intersect can be neglected.

Vavakin and Salganik (1978) found the effective compliances for an orthotropic plate with a set of non-interacting cracks aligned to one of the symmetry axes of the material. Generalising their formula one obtains the effective compliances for an orthotropic plate with two sets of non-interacting cracks of concentrations Ω_1 and Ω_2 normal to axes x_1 and x_2 (in a coordinate system x_1, x_2 aligned to the symmetry axes of the material):

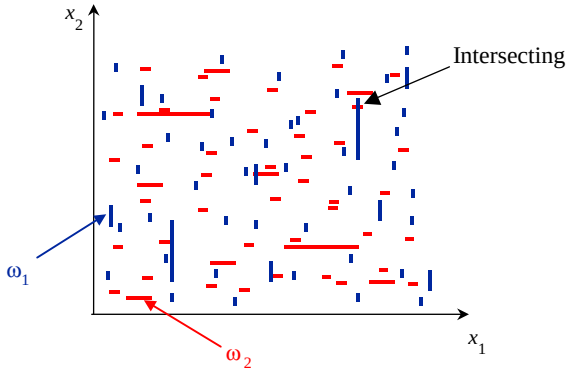

Fig. 4. A plate with two mutually orthogonal sets of cracks with the concentration factors ω_1 and ω_2 respectively. In the asymptotics $\omega\to 0$ ($\Omega_t = \text{const}$) occurrences of the crack intersection (indicated by an arrow) are rare and their influence can be neglected.

$$\begin{cases} A_{11} = A_{11}^{\mathrm{m}} + \dfrac{\pi}{4}\Omega_1\sqrt{A_{11}^{\mathrm{m}}(2A_{12}^{\mathrm{m}} + A_{66}^{\mathrm{m}} + 2\sqrt{A_{11}^{\mathrm{m}}A_{22}^{\mathrm{m}}})} \\ A_{22} = A_{22}^{\mathrm{m}} + \dfrac{\pi}{4}\Omega_2\sqrt{A_{22}^{\mathrm{m}}(2A_{12}^{\mathrm{m}} + A_{66}^{\mathrm{m}} + 2\sqrt{A_{11}^{\mathrm{m}}A_{22}^{\mathrm{m}}})} \\ A_{66} = A_{66}^{\mathrm{m}} + \dfrac{\pi}{4}\Omega_1\sqrt{A_{22}^{\mathrm{m}}(2A_{12}^{\mathrm{m}} + A_{66}^{\mathrm{m}} + 2\sqrt{A_{11}^{\mathrm{m}}A_{22}^{\mathrm{m}}})} + \dfrac{\pi}{4}\Omega_2\sqrt{A_{11}^{\mathrm{m}}(2A_{12}^{\mathrm{m}} + A_{66}^{\mathrm{m}} + 2\sqrt{A_{11}^{\mathrm{m}}A_{22}^{\mathrm{m}}})} \\ A_{12} = A_{12}^{\mathrm{m}} \end{cases} \tag{46}$$

Here A_{11}^{m}, A_{22}^{m}, A_{12}^{m}, A_{66}^{m} are the compliances of the material, such that the Hook's law has a form: $\varepsilon_{11} = A_{11}{}^{m}\sigma_{11} + A_{12}{}^{m}\sigma_{22}$, $\varepsilon_{22} = A_{12}{}^{m}\sigma_{11} + A_{22}{}^{m}\sigma_{22}$, $\varepsilon_{12} = 1/2\ A_{66}{}^{m}\sigma_{12}$.

The scaling equations can be obtained by replacing Ω_1 and Ω_2 with ω_1 and ω_2, bringing A_{ii}^{m} to the left-hand sides and then replacing $A_{ii} - A_{ii}^{\mathrm{m}}$ with βa_{ii} and, finally, replacing A_{ii}^{m} with a_{ii} in the remaining parts. The last equation in (46) reads $\beta a_{12} = 0$ such that $a_{12} = 0$. The first three equations yield the following scaling equations

$$\begin{cases} \beta a_{11} = \dfrac{\pi}{4}\omega_1\sqrt{a_{11}(a_{66} + 2\sqrt{a_{11}a_{22}})} \\ \beta a_{22} = \dfrac{\pi}{4}\omega_2\sqrt{a_{22}(a_{66} + 2\sqrt{a_{11}a_{22}})} \\ \beta a_{66} = \dfrac{\pi}{4}\omega_2\sqrt{a_{11}(a_{66} + 2\sqrt{a_{11}a_{22}})} + \dfrac{\pi}{4}\omega_1\sqrt{a_{22}(a_{66} + 2\sqrt{a_{11}a_{22}})} \end{cases} \tag{47}$$

The only solution of these equations produces the following scaling laws:

$$\begin{aligned} &A_{ij} = a_{ij}H^{\beta}, \quad \beta = \frac{\pi}{2}\sqrt{\omega_1\omega_2} \\ &a_{22} = a_{11}\left(\frac{\omega_2}{\omega_1}\right)^2, \quad a_{66} = 2a_{11}\left(\frac{\omega_2}{\omega_1}\right), \quad a_{12} = 0 \end{aligned} \tag{48}$$

It should be noted that as the concentration of one of the sets vanishes, say $\omega_2\to 0$, the exponent and all compliances but the normalising a_{11} vanish. Thus the material degenerates to the one infinitely stiff in all directions but one.

7.9. Remarks

7.9.1. Scaling laws for other effective properties

The described methodology can equally be used for determining scaling laws for other transport properties such as the thermal and elastic conductivity, diffusion, etc. As an example consider the determination

of a scaling law for effective viscosity of a fluid with multiscale distribution of rigid spheres. The effective dynamic viscosity η of a fluid of viscosity η_m with non-interacting rigid spheres has been found by Einstein (e.g., Landau and Lifshitz, 1987):

$$\eta = \eta_m\left[1+\frac{5}{2}c\right] \tag{49}$$

where c is the volumetric fraction of the spheres; its expression via the introduced above concentration reads $c=(4\pi/3)v$. The scaling equation reads: $\alpha\eta = \frac{10\pi}{3}w\eta$. Its solution produces the scaling law for viscous fluid with self-similar distribution of rigid spherical particles

$$\eta = \eta H^{\alpha}, \quad \alpha = \frac{10\pi}{3}w \tag{50}$$

7.9.2. On the correspondence between the scaling exponents for elastic moduli and fractal dimensions

There have been attempts to relate mechanical properties of a material to its fractal dimension which is a geometric property (e.g., Balankin, 1992; Bazant, 1993; Mosolov, 1993; Carpinteri, 1994; Cherepanov et al., 1995). Amongst the materials considered, the material with pores has a fractal dimension less than three. Indeed the box counting dimension of the material with self-similar distribution of spherical pores is $D=3-\frac{4\pi}{3}w$ This corresponds to the scaling exponent α of the elastic modulus (30) $D=3+\frac{\alpha}{2}$.

Materials with cracks have fractal dimension exactly equal to three since cracks have no internal volume. Thus the fractal dimension is independent of the distribution of inhomogeneities, while the scaling exponents are determined by the concentration factors w. Thus generally there is no relationship between the dimension and the mechanical scaling laws.

7.9.3. Values of Poisson's ratio

Poisson's ratio being a bounded quantity has a zero exponent and for that reason remains constant across the scales. A reason behind the constant values of Poisson's ratio is that a material with self-similar distribution of inhomogeneities formally possesses infinite total concentration of inhomogeneities at any scale. From this point of view, the values of Poisson's ratios are limiting values as the concentration tends to infinity. Interestingly, for the case of cracks the Poisson's ratio vanishes, while for pores or rigid inclusions it is 0.2 (in 3D). One can expect that randomly oriented ellipsoidal pores or rigid inclusions should yield intermediate values.

8. Scaling of stress fields

The stress field being a tensorial field should, according to Section 4, scale with a scalar exponent which in general case can depend upon the location. Thus

$$\sigma_{ij}^{H}(\mathbf{x}) = s_{ij}(\mathbf{x})H^{\chi(\mathbf{x})} \tag{51}$$

In each continuum, the stress field must satisfy the equations of equilibrium which, in the case of the absence of body forces, read $\sigma_{ij,i}^{H}(\mathbf{x})=0$, $\mathbf{x}\in F^H$, where F^H is the subset of the H-continuum that covers the points of the original fractal material F. Substituting (51) into the equations of equilibrium one has for all scales

$$s_{ij,i}(\mathbf{x}) + s_{ij}(\mathbf{x})\chi_{,i}(\mathbf{x}) = 0, \quad \mathbf{x}\in F \tag{52}$$

We will call the material *strongly self-similar* if $\chi_{,i}=0$ such that $\chi=\text{const}$. In this case both the stress field $\sigma_{ij}^{H}(\mathbf{x})$ and its volumetric average $\langle\sigma_{ij}^{H}\rangle$ scale with the same exponent

$$\sigma_{ij}^{H}(\mathbf{x}) = s_{ij}(\mathbf{x})H^{\chi}, \quad \langle\sigma_{ij}^{H}\rangle = \langle s_{ij}\rangle H^{\chi} \tag{53}$$

Since, as pointed out by Carpinteri (1994), the average stress scales as $\langle\sigma_{ij}^{H}\rangle \propto H^{D_s-2}$ where D_s is the sectional fractal dimension of the material, $\chi = D_s-2$. Thus, the Carpinteri's scaling law corresponds to the *strongly self-similar* materials. It should also be noted that in the considered theory the stresses have conventional units since the stress belongs to a continuum, while the scaling only corresponds to the transition from one continuum to another.

9. Self-similar stress concentrators

When the self-similar material has stress concentrators without characteristic length, such as edge or screw dislocations, concentrated force or semi-infinite crack, the induced stress fields will keep the power scaling law of transition between the continua. Suppose that the material is isotropic, i.e. the prefactors are isotropic (the scaling law is always isotropic, as explained in Section 3).

9.1. Dislocations

Consider a 2-D fractal material (e.g., a material with self similar distribution of inhomogeneities, such for instance as pores or cracks) and place the dislocation at the origin of a polar coordinate frame (r,φ). Let the Burgers vector be, for instance, $\boldsymbol{b} = (b,0)$. Then in an H-continuum, in a sufficiently small vicinity of the origin in which the probability of intersection with an inhomogeneity can be neglected, the stress components have the form (e.g., Landau and Lifshitz, 1959):

$$\sigma_{rr}^{H} = \sigma_{\varphi\varphi}^{H} = -\frac{bE(H)}{4\pi(1-\nu^2)}\frac{\sin\varphi}{r}, \quad \sigma_{r\varphi}^{H} = \frac{bE(H)}{4\pi(1-\nu^2)}\frac{\cos\varphi}{r} \tag{54}$$

where $E(H)\propto H^{\alpha}$, $\nu = \text{const}$ are the moduli of the H-continuum.

The radius of the vicinity of the origin where (54) is valid is obviously proportional to H. On the other hand, H determines the resolution of the corresponding continuum. Therefore, as $r\to 0$ one has to transfer to finer and finer scale, with $H\propto r\to 0$. Then, according to (54) one will observe an apparent non-conventional stress concentration:

$$\sigma \propto r^{\alpha-1} \tag{55}$$

One should emphasise that this is not a real stress concentration since the limiting transition $r\to 0$ involves transition through r-continua, while within each continuum the stress concentration is conventional, r^{-1}.

A similar situation can be observed for a screw dislocation with a Burgers vector b. The stress distribution in a cylindrical coordinate frame (r,φ,z) is (e.g., Landau and Lifshitz, 1959):

$$\sigma_{z\varphi}^{H} = \frac{bE(H)}{4\pi(1-\nu^2)}\frac{1}{r} \tag{56}$$

Again, within each continuum the stress concentration has a conventional r^{-1} singularity, while the apparent stress concentration across the scales is $\sigma\propto r^{\alpha-1}$.

9.2. Concentrated force

In an H-continuum in a sufficiently small vicinity of the origin the stress distribution generated by a concentrated force F directed along z axis of a cylindrical coordinate frame (r,ϕ,z) has the form (e.g., Timoshenko and Goodier, 1970):

$$\sigma_{rr}^H = \frac{F}{8\pi(1-\nu)}\left[(1-2\nu)\frac{z}{R^3} - 3\frac{r^2 z}{R^5}\right], \quad \sigma_{\varphi\varphi}^H = \frac{F}{8\pi(1-\nu)}(1-2\nu)\frac{z}{R^3}$$
$$\sigma_{zz}^H = -\frac{F}{8\pi(1-\nu)}\left[(1-2\nu)\frac{z}{R^3} + 3\frac{z^3}{R^5}\right], \quad \sigma_{rz}^H = -\frac{F}{8\pi(1-\nu)}\left[(1-2\nu)\frac{z}{R^3} + 3\frac{rz^2}{R^5}\right] \tag{57}$$

where $R^2 = r^2 + z^2$.

Here the stress distribution is independent of the Young's modulus, hence the apparent stress singularity coincides with the conventional one, R^{-2}.

9.3. Semi-infinite crack

Consider a 2-D fractal material and place the tip of a semi-infinite crack at the origin of a polar coordinate frame (r, ϕ). Let the crack be loaded in a self-similar manner, i.e. without introducing a characteristic length. (This can for instance be accomplished by considering a finite crack uniformly loaded and then tending its length to infinity and the load to zero such that the stress intensity factors remain constant.) Then in an H-continuum, in a sufficiently small vicinity of the crack tip in which the probability of intersection with an inhomogeneity can be neglected, the components of singular stress have the form (e.g., Sih and Liebowitz, 1968):

$$\sigma_{ij}^H(d,\theta) = \frac{K_{\mathrm{I}}^H f_{ij}(\varphi) + K_{\mathrm{II}}^H g_{ij}(\varphi)}{\sqrt{2\pi r}}, \quad i,j = 1,2 \tag{58}$$

$$f_{11} = \left(1 - \sin\tfrac{\varphi}{2}\cdot\sin\tfrac{3\varphi}{2}\right)\cos\tfrac{\varphi}{2}$$
$$f_{22} = \left(1 + \sin\tfrac{\varphi}{2}\cdot\sin\tfrac{3\varphi}{2}\right)\cos\tfrac{\varphi}{2}$$
$$f_{12} = \sin\tfrac{\varphi}{2}\cdot\cos\tfrac{\varphi}{2}\cdot\cos\tfrac{3\varphi}{2}$$
$$g_{11} = -\left(2 + \cos\tfrac{\varphi}{2}\cdot\cos\tfrac{3\varphi}{2}\right)\sin\tfrac{\varphi}{2}$$
$$g_{22} = \sin\tfrac{\varphi}{2}\cdot\cos\tfrac{\varphi}{2}\cdot\cos\tfrac{3\varphi}{2}$$
$$g_{12} = \left(1 - \sin\tfrac{\varphi}{2}\cdot\sin\tfrac{3\varphi}{2}\right)\cos\tfrac{\varphi}{2}$$

where K_{I}^H and K_{II}^H are the stress intensity factors (SIFs) of Modes I and II respectively,

Although expression (58) formally does not contain the Young's modulus, the interaction between the crack and the inhomogeneities leads to dependence of the stress intensity factors upon the scale. This dependence, in the case of the self-similar distributions of inhomogeneities in narrow sense can be obtained as follows.

Suppose that a crack is placed in a continuum material with added inhomogeneities. We compare two situations: the crack in the original material without microstructure and the crack in an effective continuum that models the material with microstructure. A crack under load is characterised by mutual displacements of the opposite faces. Placing the crack in the effective medium affects the displacement distribution. The same displacement distribution can also be achieved for the crack in the original material if certain equivalent tractions are applied to its faces. These equivalent tractions represent an average effect of interaction with the inhomogeneities, the averaging being conducted over all positions of the inhomogeneities. The equivalent tractions can be found by comparing the mutual displacements of the crack in the effective medium with those of the crack in the original material. Then by considering the crack in the original material

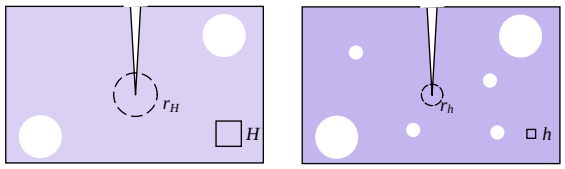

Fig. 5. A tip of semi-infinite crack in H-continuum (a) and h-continuum (b). In both cases the vicinity of the crack tip is chosen small enough to ensure that the probability of an inhomogeneity getting close to it is negligible.

to be loaded by a superposition of the original load and the equivalent tractions, the average SIFs [6] can be determined. This is equivalent to the following procedure (see Dyskin, 2002a for details): (i) compute the displacement discontinuity at the crack, assuming it to be in the effective medium under the original load and then; (ii) compute the SIFs from the obtained displacement discontinuity considering the crack contour as being in the original material. (This method was verified by Dyskin, 1985, for the 2-D elastic problem of interaction between a crack and parallel microcracks by the comparison with the results of the explicit calculation of the pairwise interactions.) In particular, the average SIF will have the form

$$\langle K_{\mathrm{I}}\rangle/K_{\mathrm{I}}^{\mathrm{m}} \sim E_{\mathrm{m}}/E \tag{59}$$

Here E_{m}, and $K_{\mathrm{I}}^{\mathrm{m}}$ are Young's modulus of and the SIF for the crack in the original material, i.e. without the interaction, E is the effective Young's modulus.

Consider now the fractal material and the crack in two continua characterised by scales H and h, $h < H$. Introduce the vicinities of crack tip of sizes $r_h \propto h$ and $r_H \propto H$ in the respective continua, Fig. 5. Let these sizes be sufficiently small such that the probability of an inhomogeneity getting close to each of these vicinities is negligible. Then formula (59) can be used with the H-continuum playing the role of the original material ($E_H \to E_{\mathrm{m}}$) and the h-continuum playing the role of the effective material ($E_h \to E$). As a result, $K_{\mathrm{I}}^h/K_{\mathrm{I}}^H \sim E_H/E_h = (H/h)^{\alpha}$, where α is the scaling component for the elastic moduli.

This leads to the conclusion that the SIF should scale as follows

$$\langle K_i(H)\rangle \sim H^{-\alpha} \tag{60}$$

where α is the scaling exponent for moduli. Since the scaling of elastic moduli is always isotropic, the scaling law (60) also holds for anisotropic materials (i.e. materials with anisotropic pre-factors).

Repeating the reasoning of Section 9.1 one can conclude that the stress distribution will have an apparent non-conventional stress singularity

$$\sigma \propto r^{-\alpha-1/2} \tag{61}$$

Once again, within each continuum the crack tip produces the conventional square root singularity. However, when the point of observation approaches the crack tip and the corresponding transition from one scale to a finer scale is considered, the stress concentration apparently assumes the power singularity with a different exponent, $\lambda = 1/2 + \alpha$. This exponent is determined by the scaling law of the elastic moduli, i.e. by the material microstructure (but not the other way around, as conventionally assumed in fractal fracture mechanics, where the stress singularity is inferred from the fractal dimension of the fracture surface, e.g.,

[6] Usually the crack propagation is supposed to be determined by extreme SIF values, Kachanov (1991). Nevertheless, in some important cases the averaging is relevant. This is for example the case of stable crack growth: a small increment in crack size does not result in failure, therefore a number of increments are required to let the crack grow sufficiently to cause a noticeable effect of deformation or fracture. Thus in considering an *average* increment of the crack growth one can express the criterion in terms of average SIFs.

Mosolov, 1993; Carpinteri, 1994; Borodich, 1997). After all, the stress concentration at the crack tip is the cause, while the crack propagation resulting in the production of the new surface is the effect.

10. Crack growth in self-similar materials

The apparent stress singularity allows the determination of the fractal dimension of the fracture surface in the spirit of e.g., paper by Gol'dstein and Mosolov (1991). The energy release rate reads

$$\mathrm{d}U/\mathrm{d}t \propto K_{\mathrm{I}}^2/E \propto t^{-3\alpha} \tag{62}$$

where t is an increment in crack propagation. Comparing (62) with the energy required to produce a fracture increment of sectional fractal dimension D_{s}, occupying unit length, $E_{\mathrm{s}}\sim t^{D_{\mathrm{s}}-1}$ one can obtain the fractal dimension of the fracture surface profile: [7]

$$D = 1 - 3\alpha = 5/2 - 3\lambda, \quad \lambda = 1/2 + \alpha \tag{63}$$

Similarly, in 3D, comparing (62) with the energy required to produce a new fracture surface of fractal dimension D, occupying unit area, $E_{\mathrm{s}}\sim t^{D_{\mathrm{s}}-2}$ one has

$$D = 2 - 3\alpha = 7/2 - 3\lambda, \quad \lambda = 1/2 + \alpha \tag{64}$$

Relations (63) and (64) are different from the usually obtained (e.g., Gol'dstein and Mosolov (1991) since scaling of the modulus has also been taken into account in (62). It should be noted that in this case the dimension of the fracture surface is determined by the scaling law of the elastic moduli, i.e., ultimately, by the material microstructure.

11. Discussion and conclusion

The main idea of the proposed approach is to model mechanics of fractal materials—the materials which are nowhere continuous—by a continuum sequence of continua of increasing scales each determined by its own size of the averaging volume element. Thus, such quantities as stress, strain, moduli, etc. are defined in each continuum in a conventional way. Then in self-similar materials the transition between the quantities belonging to different continua is represented by power laws with exponents determined by the microstructure.

Physical laws must be invariant with respect to coordinate rotations, hence the physical quantities must have the tensorial property, i.e., their components undergo certain linear transformations as the coordinate frame rotates. In therms of self-similar scaling, this is a very restrictive property because of linear independence of power functions with different exponents. As a result, the scaling laws for tensors are always isotropic (the same exponent for all non-zero components) with only prefactors capable of accounting for anisotropy. This allows the determination of the scaling laws for effective elastic characteristics in materials with self-similar distributions of pores, cracks and rigid inclusions. The obtained scaling exponents do not necessary relate to the material fractal dimension.

The scaling exponent for the stress field is generally non-uniform. However, in the case of strong self-similarity it is uniform, coincides with the one for the average (nominal) stress and is controlled by the sectional fractal dimension of the material.

The stress concentrations produced in a sufficiently small vicinity of the kernel of a linear dislocation or the point of application of a concentrated force have, within each continuum, conventional singularities.

[7] This analysis was proposed by Dyskin (2002b), however, there was a misprint in the corresponding formula of that paper.

However approaching the dislocation kernel also involves transition to finer continua which leads to apparent non-conventional singularity $\sigma \propto H^{\alpha-1}$, where α is the scaling exponent for the elastic moduli. Similar situation can be observed in a vicinity of a tip of a semi-infinite crack. In this case the apparent non-conventional singularity is caused by the interaction between the crack and microstructural elements of the material. If the crack propagates, the exponent for this singularity determines the fractal dimension of the future fracture surface.

The above shows that the self-similarity is such a restrictive property that non-trivial results are attainable in quite general situations. This suggests that not all real materials can be represented as self-similar even if some quantities scale close to the power law. It is important to determine the necessary and sufficient conditions of self-similar modelling, its accuracy and to identify the classes of systems for which this modelling is applicable.

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Appendix A. Linear independence of power functions

Lemma. *A set of power functions* $\{x^{a_i}\}$, $i = 1, \ldots, n$ *with different exponents* a_i *is linearly independent.*

Proof. When a_i are integer this property is well known.Consider the general case, when a_i are arbitrary real numbers. The linear independence means that the equation

$$\sum_{i=1}^{n} C_i x^{a_i} = 0 \tag{A.1}$$

can have only trivial solutions, $C_i = 0$. In order to demonstrate this let multiply (A.1) by x^{b-1} and integrate the sum over $(0,1)$ assuming that $b \geqslant \frac{1}{2} - \min_i a_i$. This gives

$$\sum_{i=1}^{n} \frac{C_i}{a_i + b} = 0 \tag{A.2}$$

By assigning different values, $b_1, \ldots, b_n$ to b one obtains from (A.2) a homogeneous system of equation with respect to C_i.

$$\sum_{i=1}^{n} \frac{C_i}{a_i + b_j} = 0, \quad j = 1, \ldots, n \tag{A.3}$$

The determinant of this system

$$D_n = \begin{vmatrix} \frac{1}{a_1 + b_1} & \cdots & \frac{1}{a_1 + b_n} \\ \cdots & \cdots & \cdots \\ \frac{1}{a_n + b_1} & \cdots & \frac{1}{a_n + b_n} \end{vmatrix} \tag{A.4}$$

can, following Achieser (1956) be computed as follows.

First, by extracting the common denominator, the denominator can be expressed as

$$D_n = \frac{P_n}{\prod_{i,j=1}^{n}(a_i + b_j)} \tag{A.5}$$

where P_n is a polynomial in $a_1, \ldots, a_n, b_1, \ldots, b_n$ of degree $n(n-1)$.

Second, one observes that this polynomial vanishes when either $a_k = a_l$ (two equal rows in the determinant) or $b_k = b_l$ (two equal columns). Therefore, it is divisible by

$$Q_n = \prod_{1\leqslant k<l\leqslant n}(a_k + a_l), \quad R_n = \prod_{1\leqslant k<l\leqslant n}(b_k + b_l) \tag{A.6}$$

Both these quantities are polynomials of degree $n(n-1)/2$. Hence,

$$P_n = \alpha_n Q_n R_n \tag{A.7}$$

where α_n is a constant independent of $a_1, \ldots, a_n, b_1, \ldots, b_n$.

The final step is the determination of α_n. Obviously, $\alpha_1 = 1$. Then, multiplying the last row in (A.5) by a_n, tending it to infinity and then tending b_n to infinity, one obtains D_{n-1}. On the other hand, substituting (A.7) into (A.5) and applying the same procedure (i.e., multiplying by a_n, tending it to infinity and then tending b_n to infinity) one obtains $a_n\, D_{n-1}/a_{n-1}$ One can conclude from here that $a_n = 1$. Finally,

$$D_n = \frac{\prod_{1\leqslant k<l\leqslant n}(a_k + a_l)\prod_{1\leqslant k<l\leqslant n}(b_k + b_l)}{\prod_{i,j=1}^{n}(a_i + b_j)} \neq 0 \tag{A.8}$$

when $a_k \neq a_l$ and $b_k \neq b_l$. The first inequality is satisfied by assumption, the second is satisfied by the choice of $b_1, \ldots, b_n$. This implies that system (A.2) has only trivial solution, which completes the proof.

One can also offer another, shorter proof. Indeed, by introducing the function

$$C(a) = \sum_{i=1}^{n} C_i \delta(a - a_i) \tag{A.9}$$

where $\delta(x)$ is Dirak's delta-function, and replacing x with $\exp(-p)$ one can rewrite (A.1) as

$$\int_{-\infty}^{\infty} C(a)\exp(-ap)\mathrm{d}a = 0 \tag{A.10}$$

The integral is the double Laplace transform of a generalised function $C(a)$. Since the Laplace transform is a one-to-one correspondence, $C(a) = 0$. This implies that $C_i = 0$, which finalises the second proof. □

Appendix B. Proof that (36) is the only solution of system (35)

Suppose there is another solution such that $v \neq 0$. The second equation in (35) can be rewritten in the form

$$C_1(v) + C_2(v) - 2C_3(v) = \frac{10vC_3(v)}{1-2v} \tag{B.1}$$

Using (34) one obtains

$$C_1(v) + C_2(v) = C_3(v)\frac{2-v}{1-v+v^2A} \tag{B.2}$$

where

$$A = \frac{k_a^2}{k^2}\left[1 - k_a^2\frac{K(k)}{E(k)}\right]\left[\frac{K(k)}{E(k)} - 1\right] \tag{B.3}$$

Substituting (B.2) into (B.1) one gets the following equation

$$\frac{v(9-8v+2vA+6v^2A)}{(1-v+v^2A)(1-2v)}=0 \tag{B.4}$$

In order to determine whether this equation has non-zero solution Eq. (B.3) should be analysed. Using the definitions complete elliptical integrals it can be directly demonstrated that

$$E(k)\geqslant k_a^2K(k),\quad K(k)\geqslant E(k) \tag{B.5}$$

Therefore, $A\geqslant 0$. On the other hand

$$\frac{1}{4}-A=\frac{k_a^2}{4k^4E^2(k)}[(1+k_a^2)E(k)-2k_a^2K(k)]^2 \tag{B.6}$$

As a result $0\leqslant A\leqslant 1/4$. Using this property it is easy to see that denominator in (B.4) cannot vanish and $9-8v+2vA+6v^2A>0$. Therefore, $v=0$ is the only solution of (B.1) and of the system (35). □

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